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LETTER TO THE EDITOR

On the Lenz vector

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Abstract. Some general considerations are applied to the Lenz vector, and a recent work on the subject is discussed.

Over the years there has been a recurring interest in the Lenz vector of the classical Kepler problem. Recently in this journal Prince and Eliezer (1981) have set out to find an associated symmetry group of point transformations for this vector, even though it is not associated via Noether's theorem with any groups of point transformations which leave the action invariant. The investigation of this question brings out a number of general considerations.

A first point which should be noted is that for any autonomous dynamical system in n degrees of freedom there exists, at least locally, and to within functional independence, $2n-1$ constants of the motion which do not involve time explicitly. For in a region (free of singularities or zeros) of the $2n$-dimensional space of positions and velocities/momenta, in which the system is represented by a flow of curves (one curve through each point), take an arbitrary $(2n-1)$-dimensional hypersurface crossing the curves, choose any $(2n-1)$ independent functions on the hypersurface, and drag each one along the curves so that it becomes a constant of the motion. (Similarly for any system, autonomous or non-autonomous, there are $2n$ constants of the motion which do involve the time.) Thus, for the Kepler problem in two or three dimensions there are respectively up to three or up to five independent constants of the motion not involving time. If further constants are not required to be functionally independent—and the components of Lenz's vector are not independent of the usual constants for the Kepler problem—then one can obviously construct suitable functions to form the components of a vector which is to be constant, and point in any chosen direction—such as the major axis. There is nothing significant merely in the existence of a conserved two- (respectively, three-) component object.

However, when an object is called a vector, that usually means more than just that it has two (three) components. The title implies a transformation law. The feeling that the Lenz vector is a vector comes from its form in three dimensions, constructed by vector operations from objects which are vectors under rotation:

$$\boldsymbol{R} = \dot{\boldsymbol{r}} \wedge \boldsymbol{L} - \mu \boldsymbol{r}/r \tag{1}$$

$$= \dot{r}^2 \boldsymbol{r} - (\dot{\boldsymbol{r}} \cdot \boldsymbol{r})\dot{\boldsymbol{r}} - \mu \boldsymbol{r}/r. \tag{2}$$

What one has in mind when one sees a formula like this is that it represents a directed object in configuration space, given in *any* of a class of coordinate systems connected by rotations, i.e. it is a Cartesian vector of conserved functions, is invariant in functional form under rotation, and it remains conserved for the transformed solution.

This is to require that (for the simpler case of two dimensions now) for some functions V_1, V_2 of the constants of the motion c_1, c_2, c_3:

$$\begin{aligned} \cos\alpha V_1 - \sin\alpha V_2 &= V_1(\bar{c}_A) \\ \sin\alpha V_1 + \cos\alpha V_2 &= V_2(\bar{c}_A) \end{aligned} \tag{3}$$

with $\bar{c}_A = c_A(\bar{x}^i(x^j))$. Thus there are two equations for the two functions, which in general provides a solution. The analogy of this argument should still hold for a general system. Thus, there still seems nothing peculiar to the Kepler problem in possessing a vector constant of the motion—though for general systems the corresponding vectors are unlikely to be as simple in form as Lenz's vector.

It is when one sets up a transformation between constants of the motion and infinitesimal invariances that a real restriction occurs. There are two very natural requirements of a transformation between symmetries and conservation laws, namely, that for a suitable class of transformations: (i) the transformed infinitesimal invariance is associated with the transformed constant; (ii) if a number of constants make up a geometric object in configuration space (e.g. the Lenz vector), and each corresponds to an infinitesimal point invariance, then the vectors of the latter make up an object in configuration space with a corresponding transformation law.

The standard method of associating infinitesimal invariances with constants of the motion does have such properties. If f is the constant of the motion and F the associated vector field, the standard transformation may be written

$$F = \omega^{-1} \circ \operatorname{grad} f \tag{4}$$

where ω is the closed two-form, given in the $2n$-dimensional space of positions and velocities by

$$\omega = \begin{pmatrix} \dfrac{\partial^2 L}{\partial x^i \partial \dot{x}^j} - \dfrac{\partial^2 L}{\partial \dot{x}^i \partial x^j} & -\dfrac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} \\ \dfrac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} & 0 \end{pmatrix}. \tag{5}$$

In the $(2n+1)$-dimensional space of time, positions and velocities, ω is given by

$$\begin{pmatrix} 0 & -\dfrac{\partial L}{\partial x^i} + \dfrac{\partial^2 L}{\partial x^i \partial \dot{x}^k}\dot{x}^k + \dfrac{\partial^2 L}{\partial t \partial \dot{x}^i} & \dfrac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^k}\dot{x}^k \\ \dfrac{\partial L}{\partial x^i} - \dfrac{\partial^2 L}{\partial x^i \partial \dot{x}^k}\dot{x}^k - \dfrac{\partial^2 L}{\partial t \partial \dot{x}^i} & \dfrac{\partial^2 L}{\partial x^i \dot{x}^j} - \dfrac{\partial^2 L}{\partial \dot{x}^i \partial x^j} & -\dfrac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} \\ -\dfrac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^k}\dot{x}^k & \dfrac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} & 0 \end{pmatrix} \tag{6}$$

and the 'ω^{-1}' in (4) is now interpreted as a mapping to equivalence classes of vector fields differing by multiples of the motion vector field (Crampin 1977). These results are obtained by transforming the canonical two-form of Hamiltonian theory back to the Lagrangian picture by the Legendre transformation; the transformation from constants to invariances is by this means expressed in a form which covers both the usual Lagrangian result (Noether's theorem), and the standard results in the Hamiltonian picture. As the author has shown elsewhere (Schafir 1981) the transformation (4) has the invariance properties mentioned above, and indeed can be characterised by them.

The Kepler problem in two dimensions admits a three-parameter group of point transformations: time translation, rotation, and the transformations generated by the vector field:

$$t\,\partial/\partial t+\tfrac{2}{3}x\,\partial/\partial x+\tfrac{2}{3}y\,\partial/\partial y-\tfrac{1}{3}\dot{x}\,\partial/\partial\dot{x}-\tfrac{1}{3}\dot{y}\,\partial/\partial\dot{y} \qquad (7)$$

(see Prince and Eliezer 1981). For the usual Lagrangian, (7) is not the image of any function under $\omega^{-1}\circ$ grad, while at the same time the symmetries corresponding to Lenz's vector are 'hidden' (i.e. they are not point transformations). Prince and Eliezer looked for an alternative transformation between constants of the motion and infinitesimal invariances which would associate both components of the Lenz vector with the vector field (7). Their method, translated into geometrical terms, is to look for constants of both motions, the dynamical motion field, and the flow generated by (7). (Thus they are imitating one feature of the usual theory, the skew symmetry of ω, which causes any function to be constant along its own associated direction.) Clearly there exist two such functions, to within functional independence; they simply choose the two which are the components of Lenz's vector.

However, there is a very high degree of arbitrariness in this result. Also, it lacks the first of the invariance properties referred to above (the second is inapplicable, since they are associating two constants of the motion with one infinitesimal invariance). For it is not the case even under the same invariances of the theory, that the transformed Lenz vector corresponds to the transformed vector field. For instance, under the finite transformations generated by (7) itself, the vector field is of course invariant; but the Lenz vector, transforming as a vector—as opposed to each component transforming as a scalar—is not invariant.

Finally, note that the invarance group of a system is initially infinite-dimensional, and is independent of any choice of a Lagrangian. The introduction of a Lagrangian gives us a mapping from the constants of the motion into, but not onto, the infinitesimal generators of invariances, for which the image vector fields generate a finite-dimensional group. For the two-dimensional Kepler problem, with the usual Lagrangian, this group is SO(3), and this fact has upon quantisation, the physical consequence of degeneracy of the spectrum (Jauch and Hill 1940). A different Lagrangian would in general give rise to a different transformation and a different finite group, while as we have seen, a transformation which does not arise from a Lagrangian is likely to lack some obviously necessary invariance properties.

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